Process Control, 3P4

Assignment 2

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Assignment objectives: gaining an excellent understanding of dynamic systems; using computer models and Laplace transforms to infer and describe the model behaviour over time.

Question 1 [10]

For each transfer function below, what can you say about the time-domain function, $H(t)$, where $0 \leq t \leq \infty$? Do not solve for $H(t)$ analytically.

Your answers should include the initial value, final value, and a description of the stability, smoothness, and oscillatory behaviour.

1. $H(s) = \dfrac{6(s+2)}{(s^2+9s+20)(s+4)}$
2. $H(s) = \dfrac{10s^2-3}{(s^2-6s+10)(s+2)}$
3. $H(s) = \dfrac{16s+5}{s^2+9}$

Solution

The solution is by inspection of the transfer functions, there is no need to actually invert to the time domain.

All functions in this question are smooth, because there are no discontinuities (e.g. steps, spikes, *etc*).

1. Find the initial and final values:

$$\begin{aligned}
\lim_{t\to 0} h(t) &= \lim_{s\to\infty} sH(s) \\
&= \lim_{s\to\infty} s\frac{6(s+2)}{(s^2+9s+20)(s+4)} \\
&= \lim_{s\to\infty} \frac{6s^2+12s}{s^3+13s^2+56s+80} \\
&= \lim_{s\to\infty} \frac{6}{s+13} \qquad \text{l'Hôpital's rule} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty} h(t) &= \lim_{s\to 0} sH(s) \\
&= \lim_{s\to 0} \frac{6(s+2)}{(s^2+9s+20)(s+4)} \cdot s \\
&= 0
\end{aligned}$$

therefore we can say this transfer function is stable.

Inspecting the terms in the denominator shows that we would have done a partial fraction expansion with a $(s+4)$ term, and a $(s^2+9s+20)$ term. This second term is expanded to $(s^2+9s+20) = (s+4)(s+5)$

In other words:

$$H(s) = \frac{6(s+2)}{(s+4)(s^2+9s+20)} = \frac{\alpha_1}{(s+4)} + \frac{\alpha_2}{(s+4)^2} + \frac{\alpha_3}{(s+5)}$$

Our goal here isn't to find the α values, but if we did, we would have had an $\alpha_1 e^{-4t}$ term, a $\alpha_2 t e^{-4t}$ term, and an $\alpha_3 e^{-5t}$ term. None of these are oscillatory terms, they are all smooth. So the response will be smooth, and non-oscillatory.

2. As with the previous question we can find the initial and final values:

$$\begin{aligned}
\lim_{t\to 0} h(t) &= \lim_{s\to\infty} sH(s) \\
&= \lim_{s\to\infty} s\frac{10s^2 - 3}{(s^2 - 6s + 10)(s+2)} \\
&= \lim_{s\to\infty} \frac{10s^3 - 3s}{s^3 - 4s^2 - 2s + 20} \\
&= \lim_{s\to\infty} \frac{30s^2 - 3}{3s^2 - 8s - 2} \qquad \text{l'Hôpital's rule is successively applied} \\
&= \lim_{s\to\infty} \frac{60s}{6s - 8} \\
&= \lim_{s\to\infty} \frac{60}{6} \\
&= 10
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty} h(t) &= \lim_{s\to 0} sH(s) \\
&= \lim_{s\to 0} s\frac{10s^2 - 3}{(s^2 - 6s + 10)(s+2)} \\
&= 0
\end{aligned}$$

therefore we can say this transfer function is stable.

Inspecting the terms in the denominator shows that we cannot find real factors for $(s^2 - 6s + 10)$; the quadratic solution involves the square root of negative numbers. High school math shows that the roots are $3 \pm i$. If you got stuck at this point, then that's OK. We will learn how to deal with these systems later on. We will see that the positive complex roots indicate unstable operation.

3. As with the previous question we can find the initial and final values:

$$\begin{aligned}
\lim_{t\to 0} h(t) &= \lim_{s\to\infty} sH(s) \\
&= \lim_{s\to\infty} s\frac{16s + 5}{s^2 + 9} \\
&= \lim_{s\to\infty} \frac{16s^2 + 5s}{s^2 + 9} \\
&= \lim_{s\to\infty} \frac{32s + 5}{2s} \qquad \text{l'Hôpital's rule is successively applied} \\
&= \lim_{s\to\infty} \frac{32}{2} \\
&= 16
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty} h(t) &= \lim_{s\to 0} sH(s) \\
&= \lim_{s\to 0} s\frac{16s + 5}{s^2 + 9} \\
&= 0
\end{aligned}$$

therefore we can say this transfer function is stable.

Again we cannot find real factors for $(s^2 + 9)$; the roots are $0 \pm 3i$. You should not have got stuck with this one though: this is related to line 14 from the table, indicating the time-domain function will have a sinusoid term.

So the response will be smooth and oscillatory.

Question 2 [4]

Find the inverse transform of:

$$Y(s) = \frac{s+2}{s(s+1)^2}$$

Solution

You may expand the transfer function using partial fraction expansion in more than one way. Here is one method:

$$\begin{aligned}
Y(s) &= \frac{s+2}{s(s+1)^2} \\
\frac{s+2}{s(s+1)^2} &= \frac{\alpha_1}{s} + \frac{\alpha_2}{(s+1)} + \frac{\alpha_3}{(s+1)^2} \\
(s+2)(s+1) &= \alpha_1(s+1)(s+1)^2 + \alpha_2(s)(s+1)^2 + \alpha_3(s)(s+1) \\
s^2+3s+2 &= \alpha_1(s^3+3s^2+3s+1) + \alpha_2(s^3+2s^2+s) + \alpha_3(s^2+s) \\
s^2+3s+2 &= (\alpha_1+\alpha_2)s^3 + (3\alpha_1+2\alpha_2+\alpha_3)s^2 + (3\alpha_1+\alpha_2+\alpha_3)s + \alpha_1 \\
\alpha_1+\alpha_2 &= 0 \\
3\alpha_1+2\alpha_2+\alpha_3 &= 1 \\
3\alpha_1+\alpha_2+\alpha_3 &= 3 \\
\alpha_1 &= 2
\end{aligned}$$

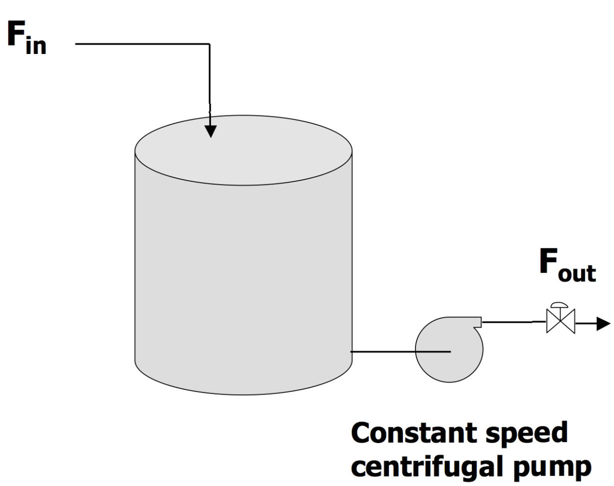
Solving simultaneously: $\alpha_1 = 2,\ \alpha_2 = -2,\ \alpha_3 = -1$

Then we can invert to obtain the time-domain solution.

$$\begin{aligned}
Y(s) &= \frac{2}{s} - \frac{2}{(s+1)} - \frac{1}{(s+1)^2} \\
y(t) &= 2 - 2e^{-t} - te^{-t}
\end{aligned}$$

Question 3 [20]

We want to determine the dynamic response of the liquid level in the open tank shown in the figure.



The flow entering is determined from an upstream process (we cannot control that flow rate). The flow out is determined by the valve percent opening, with a constant speed, centrifugal pump. (The level in the tank in this system does not significantly affect the flow out because of the high pressure supplied by the pump.)

Some data and details are given in the following:

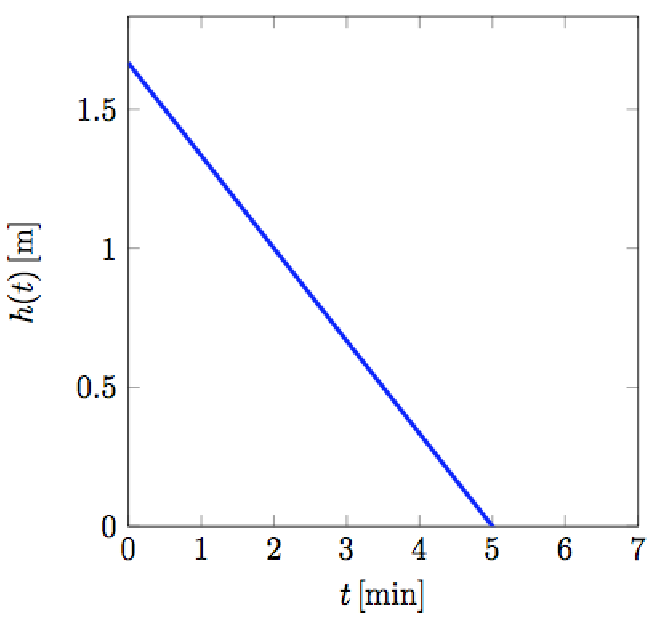
- Initial flow in = $2.0\,\text{m}^3.\text{min}^{-1}$
- Initial flow out = $2.0\,\text{m}^3.\text{min}^{-1}$
- Total tank volume (when full) = $10.0\,\text{m}^3$
- Initial tank level = 50% full
- Tank cross-sectional area = $3.0\,\text{m}^2$
- Assume that the tank is initially at steady state with the flows in and out equal.

A step change is introduced in the inlet flow; it changes from $2.0\,\text{m}^3.\text{min}^{-1}$ to $1.0\,\text{m}^3.\text{min}^{-1}$. No change is made to the valve opening.

1. Without using a computer or analytically derived expression, draw the plot of how you expect the liquid level to change once the step change is made. [There are no grades for this part of the question, so don't modify your graph later on].
2. Determine the actual dynamic response of the liquid level using a computer simulation.
3. Based on your graph and numerical results, what do you conclude about whether the level should be controlled using the feedback principle by adjusting the valve in the output after the pump?
4. The liquid is water at 25°C, and we want to measure the level within $\pm$ 1 percent of its true value. What level sensor would you recommend? Give a brief explanation with a citation for information used in your answer. [Hint: use information available at http://pc-education.mcmaster.ca/ and follow links to the Instrumentation section].
5. The flow rate out of the tank must be measured with an accuracy of $\pm$ 1 percent from about 0.25 to $2.0\,\text{m}^3.\text{min}^{-1}$. What flow sensor would you recommend? Give a brief explanation with a citation for information used in your answer.

Solution

1. The system starts at steady state (flow in = flow out). If the inlet flow drops to a new, lower value while the outlet flow remains constant, then the outlet flow exceeds the inlet flow, so the tank level drops. Your graph should show a decreasing trend, though you may not have had a linear decrease.
2. The actual response is shown below:



3. A variable can be controlled only if there is a causal relationship from the manipulated variable to the controlled variable. We know that the outlet flow (valve position) will affect the level, so yes, the level can be adjusted with feedback control when manipulating the outlet valve position.
4. From section 2.4 of that website we can conclude that all level sensors are appropriate, except for a float sensor (since our tank is at least 3 meters in height). Our fluid (water) at ambient temperature will meet the constant density assumption, so we would likely select a differential pressure sensor.
5. Some of the sensors listed in section 2.2 of the website can be eliminated: orifice, nozzle, elbow and annubar. A turbine, vortex shedding and positive displacement sensor may be selected. For example, a vortex shedding sensor should be used to determine the flow measurement, because it has the required range capability to deal with the range of flows and it is accurate enough to provide an appropriate reading.

Question 4 [12]

A heat exchanger system has a differential equation below that shows how the temperature changes with a change in flow rate, $q(t)$.

$$2\frac{dT}{dt} = -T + 5q$$

The temperature, T and q are in deviation form already.

1. $q(t)$ is changed from 0 to 2.0 at time $t = 0$. Sketch the response up to the point where the temperature reaches its new steady state.
2. What is this steady state?
3. How long does it take to reach within 0.1 degree of the new steady state.
4. If instead $q(t)$ is changed from 0 to 4.0 at time $t = 0$. Sketch the response again, and report the new steady state value of the temperature.
5. Superimpose the plots from part 1 and part 4 of this question and explain why the curves have the same shape.

Solution

1. As the variables are in deviation form already, assuming deviations around steady state, then we will emphasize the deviations with dashed notation: $2\dfrac{dT'}{dt} = -T' + 5q'$

   Taking the Laplace transform: $2sT'(s) + T'(t=0) = -T'(s) + 5Q'(s)$, where $T'(t=0) = 0$, based on our prior assumptions.

   Simplifying: $T'(s)\,(2s+1) = 5Q'(s)$

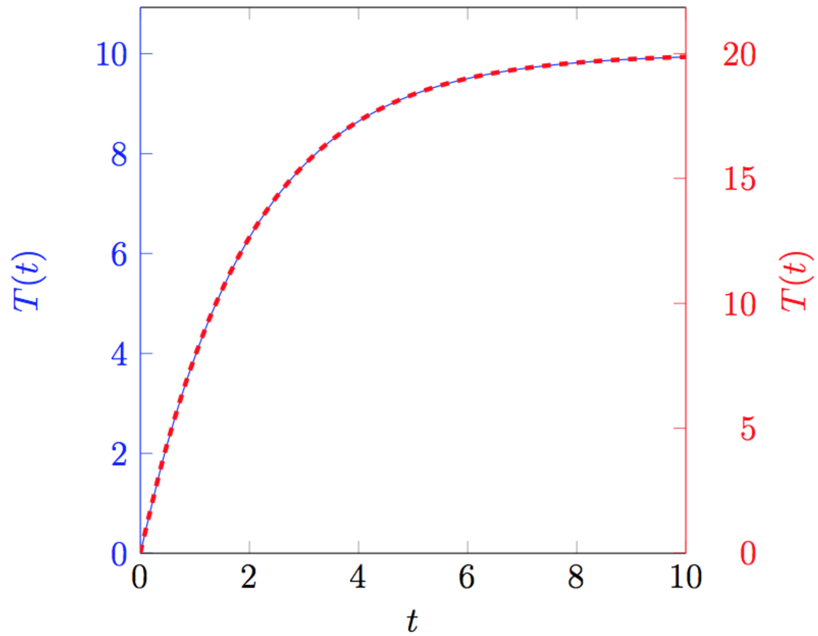
   The suitable Laplace transform form for the change in $q(t)$ is: $Q'(s) = \dfrac{2}{s}$, so we can then write that: $T'(s)\,(2s+1) = \dfrac{10}{s}$

   $$\begin{aligned} T'(s) &= \frac{10}{s\,(2s+1)} \\ T'(t) &= 10\left(1 - e^{-t/2}\right) \qquad \text{from line 13 in the Laplace transform table} \end{aligned}$$

   So a sketch for this response is combined with the answer to part 5, and shown below.
2. The steady state value is 10 units; which can be found from the analytical solution above, or using the final value theorem on the $T'(s)$ expression above.
3. Since it approaches the steady state from below, we wish to obtain the time when $T'(t) = 10 - 0.1 = 9.9$. Using the analytical expression above, this is $-2 * \ln(1 - \frac{9.9}{10}) = 9.21$ minutes.
4. Using a similar calculation, the new steady state will be 20 units.

5. The TA (Mudassir) has created an interesting plot, shown below where the left blue axis is the result for a 2 unit step input, and the right red axis is for the 4 unit step input. Even if you did not overlap them like this, you would notice the two plots have the identical response shape, but end up at different final values.

   The reason is because the denominator of the transfer function $T'(s) = \dfrac{5P}{s\,(2s+1)}$ is identical for a step input of size P units. The denominator is primarily responsible for the shape of the response, while the numerator is responsible for the end point of the response. The numerator gain magnifies or shrinks the response.



Question 5 [15]

In this question, you will reinforce the material learned in Chemical Engineering `3E04`. The system that you will consider is given in Figure 4. It is a CSTR where the reaction A $\longrightarrow$ B is occurring. You do not have to derive the component material balance, i.e., the differential equation in the figure.

1. First, you must determine the *initial* value for the reactor concentration (which is the same as the concentration leaving the reactor), C_A. The system is initially at steady state, with $C_{A,0} = 0.5\,\text{mol.m}^{-3}$.

   You may solve this part analytically or numerically. *Hint*: you will solve a quadratic equation.

2. Starting from this steady state, determine the dynamic response of the concentration C_A in the reactor if a step increase of $C_{A,0}$ is made from $C_{A,0} = 0.5\,\text{mol.m}^{-3}$ to $C_{A,0} = 1.0\,\text{mol.m}^{-3}$.

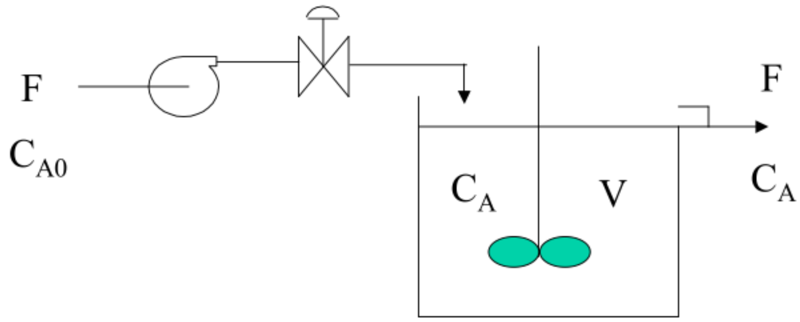
   For example, the above occurs when your manager has asked you to increase the concentration of species B that you are producing.

3. How long does it take for $C_A(t)$ to reach a new steady state?

4. You want to reach the new steady state faster. Explain how you can decrease the time taken. Perform a new simulation with the change(s) implemented, and superimpose the original time-domain plot from part 2 to show how much faster your response is.

Work ahead: in the next tutorial and assignment you will create deviation variables C'_A for outlet concentration, and C'_{A0} for inlet concentration. Create a linearized model, invert it with the Laplace transform, and compare the linearized (approximate) model to the actual model.

Solution



$$r_A = \frac{-k_1 C_A}{(1 + k_2 C_A)}$$

$$V \frac{dC_A}{dt} = F(C_{A0} - C_A) + (V)(r_A)$$

$$V = 2.0 m^3 \quad F = 1.0 m^3 / h \quad k_1 = 1.0 h^{-1} \quad k_2 = 1.0 m^3 / mole$$

1. To derive the starting value for C_A we set the differential equation to zero. Note, the starting value for $C_A \neq C_{A,0}$. The $C_{A,0}$ is the reactor inlet concentration, which is very different (a basic concept you must know from your Reactor Design course).

   Set the ODE to zero and you will obtain a quadratic expression, which on substituting in the values gives the final equation:

   $$\begin{aligned}(k_2 F)C_A^2 + (Vk_1 - k_2 F C_{A,0} + F)\, C_A - C_{A,0} F &= 0 \\ C_A^2 + 2.5 C_A - 0.5 &= 0\end{aligned}$$

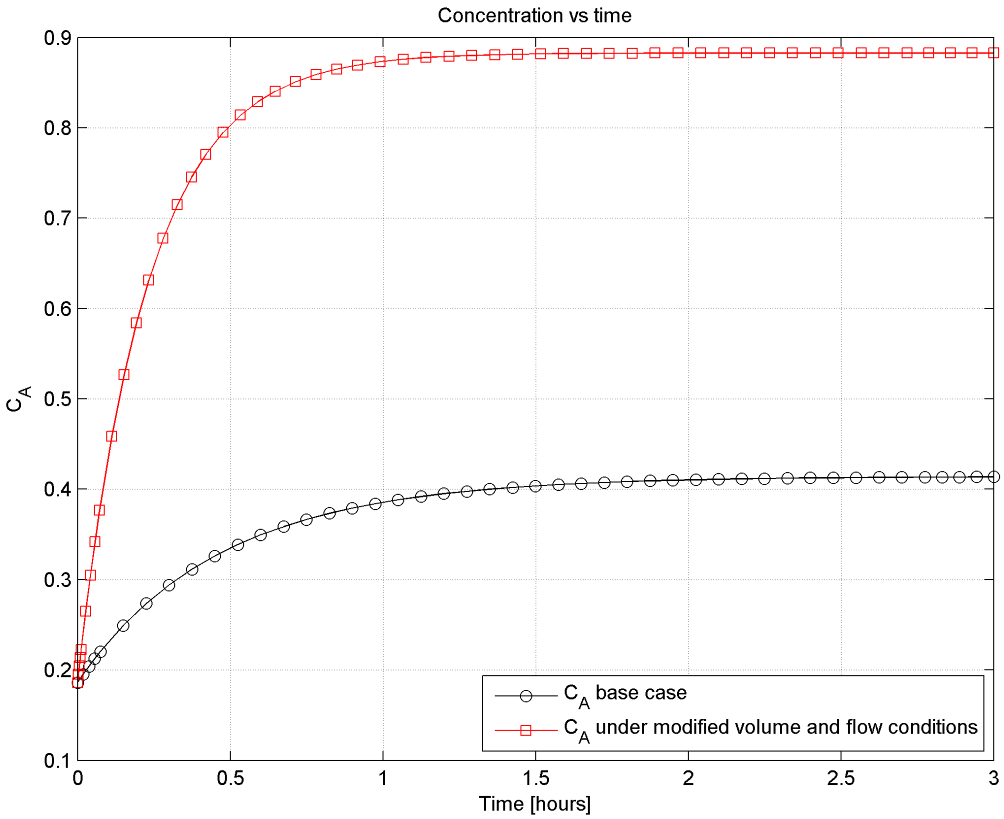
   The positive root of this quadratic yields $C_A = 0.186\,\text{mol.m}^{-3}$.

2. The response is shown below.
3. It takes between 2 to 3 hours to reach the new steady state, which is quite long.
4. A shorter time can be achieved by modifying the system and its operation. For example:

   - decrease the volume in the reactor (this is feasible in most situations by operating the CSTR with less material in it)
   - increase in the inlet flow rate

   Below we do both and show the response superimposed on the the original response. Later in the course we will design a feedback controller that will manipulate the inlet flow to achieve the desired setpoint change.

   Note that the use of flow rate isn't necessarily the best choice of manipulated variable, and isn't always possible in practice. For 2 reasons: inlet flow is often beyond our control, we have to accept the inlet flow we receive. Secondly, a faster inlet means the material in the tank has less time to react, so you increase your amount of unreacted A leaving, leading to increased separation costs downstream.



ODE_driver_assgn2_Q5.m

```
% Integrating ODEs
% ------------------

% The independent variable always requires an initial and final value:
indep_start = 0.0; % hours
indep_final = 3.0; % hours

% Set initial condition(s): for integrating variables (dependent variables)
CA_depnt_zero = 0.1861;     % i.e. CA(t=0) = 0.186

IC = [CA_depnt_zero];

% Integrate the ODE(s):
[indep1, depnt1] = ode45(@assign2_Q5_part2, [indep_start, indep_final], IC);
[indep2, depnt2] = ode45(@assign2_Q5_part4, [indep_start, indep_final], IC);

% Plot the results:
clf;
plot(indep1, depnt1(:,1), 'ko-')
grid('on')
hold('on')
plot(indep2, depnt2(:,1), 'rs-')

xlabel('Time [hours]')
ylabel('C_A')
legend({'C_A base case', 'C_A under modified volume and flow conditions'}, ...
    'Location', 'SouthEast' )
```

```
title('Concentration vs time')
print('-dpng', 'assign2_Q5.png', '-r300')
```

assign2_Q5_part2.m

```
function d_depnt__d_indep = assign2_Q5_part2(indep, depnt)

% Dynamic balance for the CSTR height

%    indep: the independent ODE variable, such as time or length or the reactor
%    depnt: a VECTOR of dependent variables
%
%    Returns:
%
%    d(depnt)
%   ---------- = a vector of ODEs
%    d(indep)

% Assign some variables for convenience of notation: one row per DEPENDENT variable
CA = depnt(1);

% Constants and other equations
V  = 2; % tank volume [m^3]
F  = 1; % inlet flow [m^3/hour]
k1 = 1; % rate constant [1/hour]
k2 = 1; % rate constant [m^3/mol]

% Step input for CA0 is made at time 0, which is when this simulation
% starts
CA0 = 1.0; % inlet concentration [mol/m3]

% Output from this ODE function must be a COLUMN vector, with n rows
% n = how many ODEs in this system?
n = numel(depnt); d_depnt__d_indep = zeros(n, 1);

% Specify every element in the vector below: 1, 2, ... n
d_depnt__d_indep(1) = F*(CA0 - CA) - V*k1*CA / (1+k2*CA);
```

assign2_Q5_part4.m

```
function d_depnt__d_indep = assign2_Q5_part4(indep, depnt)

% Dynamic balance for the CSTR height

%    indep: the independent ODE variable, such as time or length or the reactor
%    depnt: a VECTOR of dependent variables
%
%    Returns:
%
%    d(depnt)
%   ---------- = a vector of ODEs
%    d(indep)

% Assign some variables for convenience of notation: one row per DEPENDENT variable
CA = depnt(1);

% Constants and other equations
V  = 0.5; % tank volume [m^3]
F  = 2.0; % inlet flow [m^3/hour]
k1 = 1; % rate constant [1/hour]
```

```
k2 = 1; % rate constant [m^3/mol]

% Step input for CA0 is made at time 0, which is when this simulation
% starts
CA0 = 1.0; % inlet concentration [mol/m3]

% Output from this ODE function must be a COLUMN vector, with n rows
% n = how many ODEs in this system?
n = numel(depnt); d_depnt__d_indep = zeros(n, 1);

% Specify every element in the vector below: 1, 2, ... n
d_depnt__d_indep(1) = F/V*(CA0 - CA) - k1*CA / (1+k2*CA);
```

END