Optimization for Chemical Engineers, 4G3

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Kevin Dunn, kevin.dunn@mcmaster.ca **McMaster University**

Note:

- No papers, other than this test and the answer booklet are allowed with you in the midterm. You will be provided with the class-sourced cheat sheet (attached on the last page, page 6).
- You may only use the standard McMaster calculator in the midterm.
- **To help us with grading, please start each question on a new page, but use both sides of each page in your booklet**.
- You may answer the questions in any order on all pages of the answer booklet.
- This exam requires that you apply the material you have learned here in 4G3 to new, unfamiliar situations, which is the level of thinking we require from students that will be graduating and working very soon.
- **Any ambiguity or lack of clarity in a question may be resolved by making suitable and justifiable assumption(s), and continuing to answer the question with that assumption(s)**.
- There are 88 marks and you have 2 hours.
- There are 6 pages on the exam (including the 1 page cheat sheet), please ensure your copy is complete.

Question 1 [12 = 2 + 6 + 1 + 3]

Quick, short answer questions. Please provide explanations where requested.

1. For a linear programming (LP) problem where the objective function is a function of the search variables (i.e. the objective function is not equal to a fixed constant), can the optimal solution to the LP be at one of the interior points? Explain. **[2]**
2. Convert the following problem to standard form, add slack variables where necessary: **[6]**:

$$\begin{array}{lrcl} \text{maximize} & 3x_1 + x_2 + x_3 & & \\ \text{subject to} & & & \\ & x_1 - 2x_2 + x_3 & \leq & 11 \\ & -4x_1 + x_2 + 2x_3 & \geq & 3 \\ & 2x_1 - x_3 & = & -1 \\ & x_1, x_3 & \geq & 0 \\ & x_2 & \leq & 0 \end{array}$$

3. During iterations of the Simplex method, do the basic or non-basic variables have their values changed to zero? **[1]**
4. You read on a website that “the marginal price is the change in the objective function for a change of $+1$ in the right-hand side of an inequality constraint”. Do you agree with this statement in general? Are there any cases where that interpretation would, strictly speaking, be incorrect? **[3]**

Question 2 [5]

You are solving an optimization problem to improve your existing process, and the solver successfully converges (solves) for an optimum solution, which is to maximize profit.

However the solution reported by your GAMS solver gives a value of profit that is lower than the profit you are currently making on your process.

Describe some things you would investigate with the model to fix this obvious problem.

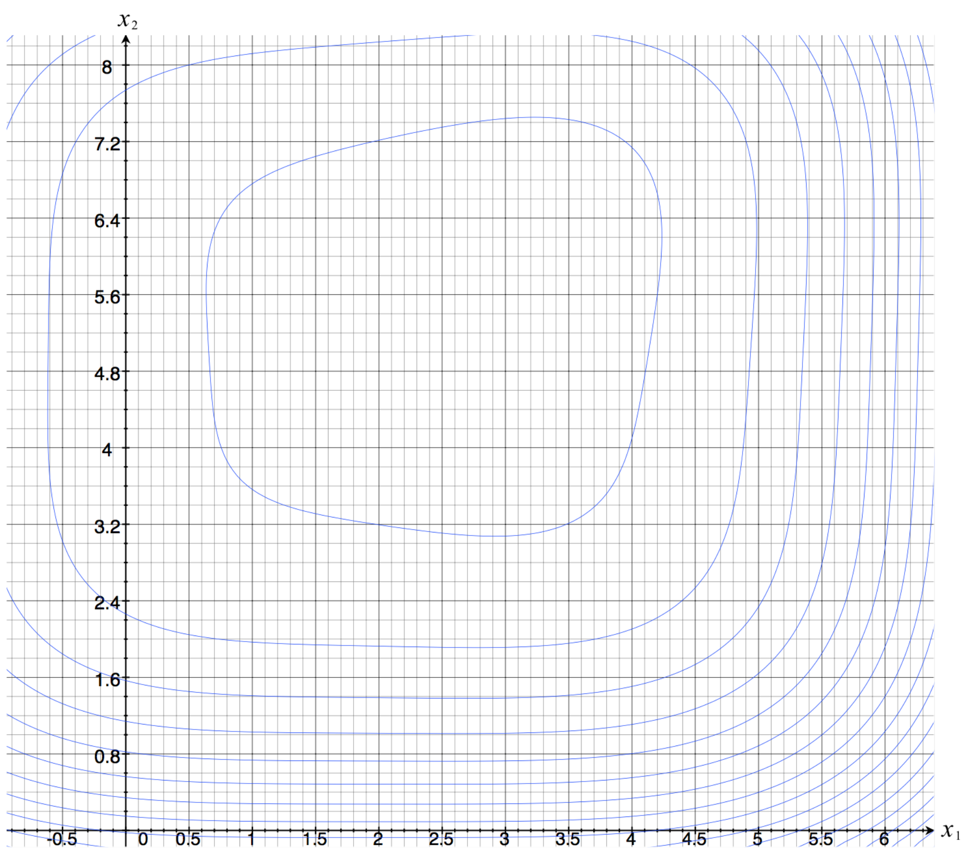
Question 3 [18 = 5 + 10 + 3]

In this question, you will investigate the behaviour of numerical optimization methods on the following function:

$$\max\ 4x_1x_2 - 5(x_1-2)^4 - 3(x_2-5)^4$$

starting from the initial point $\mathbf{x}^0 = [1,3]$. To assist you, a contour plot is attached, and the optimum is somewhere in the plot.

1. Determine the search direction that would be used for the steepest decent method. **[5]**
2. Perform a line search and determine the values of x_1 and x_2 that will be used for the next iteration, $\mathbf{x}^1$, based on the line search result. In your answer, explain how you select the values of step size α, and how you obtain the line-search optimum. **[10]**
3. Explain why a line search does **not** need to be solved precisely. **[3]**



Question 4 [[26 = 2 + 2 + 3 + 2 + 3 + 5 + 5 + 4]]

Your company is implementing an engineering design and 3 types of employees are available.

- Outsource hours from another foreign company (`Outside`) at $7 per hour; and/or
- Newly graduated (`Grad`) students at $12 per hour; and/or
- Unlimited hours of professional engineers (`Pro`) at $32 per hour.

The project is not based in Canada, but these salaries are reasonable for the country where the work is taking place.

The full project would take professional engineers at least 1,000 hours (professional-equivalent hours). Graduated students could do the work, but are only 40% as productive, and outsourced workers are only 25% as productive.

The company supervisor has only 160 fixed hours that must be allocated to this project, and it is known from experience that outside engineers require more supervision than graduates, and graduates more than professionals. The supervisor has used rates of 0.2 hours of time per hour of outside engineering time, 0.15 hours of time per hour of graduated engineer, and only 0.05 hour of time required per hour of professional engineer.

The GAMS code that implements the above problem is given below.

```
SETS
   I 'Resources' / Outside, Grad, Pro /;

PARAMETER
   C(I) 'Hourly Rate ($/hr)'
   / Outside  7.0
     Grad    12.0
     Pro     32.0 /
   S(I) 'Supervisory Rate (hr/hr)'
   / Outside  0.20
     Grad     0.15
     Pro      0.05 /
   P(I) 'Productivity fraction'
   / Outside  0.25
     Grad     0.40
     Pro      1.00 /;

SCALARS
   Project_max_span    'Full project time (hr)'      / 1000.0 /
   Graduate_max_time   'Max grad student time (hr)'  /  500.0 /
   Supervisor_max_time 'Available time (hr)'         /  160.0 / ;

VARIABLES
   X(I) 'Contracted Time (hr)'
   Z    'Cost ($)';

POSITIVE VARIABLE X;

EQUATIONS
   Cost       'Project cost'
   Work       'Required work'
   Grad_limit 'Graduate work limit'
   Supervise  'Supervisor availability';

Cost..        Z =E= SUM(I, C(I)*X(I));
Work..        SUM(I, P(I)*X(I)) =G= Project_max_span;
Grad_limit..          X('Grad') =L= Graduate_max_time;
Supervise..   SUM(I, S(I)*X(I)) =L= Supervisor_max_time;

MODEL Project / ALL /;

Project.OPTFILE=1;
SOLVE Project USING LP minimizing Z;
```

The following GAMS report was generated when solving the above model, using `CPLEX` as the LP solver:

```
---- VAR X  Contracted Time (hr)

           LOWER     LEVEL     UPPER    MARGINAL
Outside      .      240.000     +INF       .
Grad         .      500.000     +INF       .
Pro          .      740.000     +INF       .
```

```

                          LOWER      LEVEL      UPPER    MARGINAL

---- VAR Z                 -INF  31360.000      +INF        .
  Z  Cost ($)

EQUATION NAME                                   LOWER      CURRENT        UPPER
-------------                                   -----      -------        -----
Cost                                             -INF            0         +INF
Work                                            306.3         1000         1900
Grad_limit                                          0          500        846.2
Supervise                                         115          160          715

VARIABLE NAME                                   LOWER      CURRENT        UPPER
-------------                                   -----      -------        -----
X(Outside)                                      6.846            7            8
X(Grad)                                          -INF           12        12.11
X(Pro)                                          31.53           32         +INF
Z                                                -INF            1         +INF

                          LOWER      LEVEL      UPPER    MARGINAL
---- EQU Cost               .          .          .         1.000
---- EQU Work           1000.000   1000.000      +INF      32.267
---- EQU Grad_limit        -INF     500.000    500.000    -0.107
---- EQU Supervise         -INF     160.000    160.000    -5.333
```

Answer each of the following questions from the results given in the GAMS report. **When reporting numerical values, please make sure to also report the correct units.**

Hint: it might be helpful to rewrite the GAMS code into a mathematical model form, but you should be comfortable reading and interpreting it, because GAMS code is so very similar to the mathematical notation you would have used.

1. Describe, in plain language, what the objective function is aiming to minimize/maximize in this problem. **[2]**
2. What is the optimum value of this objective function at the optimum? **[2]**
3. How many hours, according to this model, should you put job postings out for (a) graduated students and (b) professional engineers? **[3]**
4. Which constraints are active at the optimum? **[2]**
5. What is the effect of an extra hour of professional-equivalent work to this project? Explain your answer. **[3]**
6. Does the supervisor's availability limit the optimal solution?

   By how much would the solution change if the supervisor could devote an extra an 50 hours, so a total of 210 hours, to supervision?

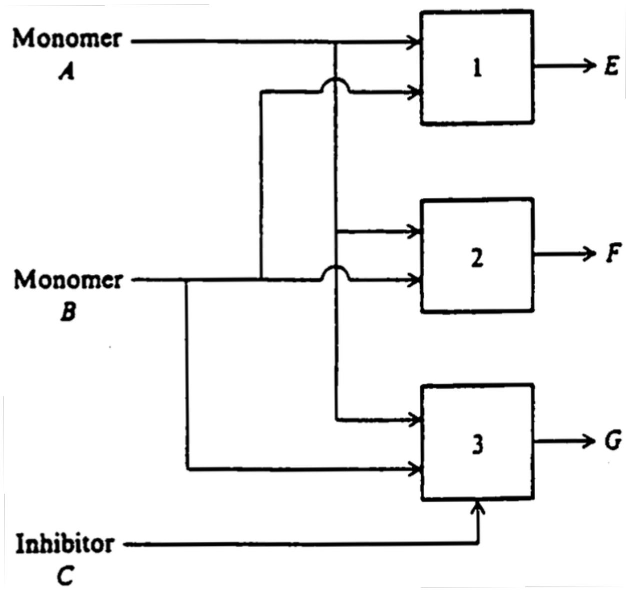
   What if the supervisor could only devote 100 hours in total to supervision? **[5 for all 3 sub-parts]**
7. An alternative option is to hire a full-time co-op student, and it is estimated that this might cost \$7,000, but it will reduce the project down to 700 professional-equivalent hours of work (instead of 1,000). However, it would cost us \$7,000 to hire that summer student and the supervisor hopes this will also reduce supervision to 135 hours (because contact time will be with one person, rather than multiple people).

   Is this a viable alternative? In particular determine (or give upper/lower bounds as best you can), the corresponding change in objective function. **[5]**
8. The outsourced (`Outside`) engineering hourly contract rate is up for negotiation. The company that provides this service wants to be payed \$7.50 per hour.

   Determine (or give upper/lower bounds as best you can) the corresponding change in cost for this. What will be the effect on the overall decision variables? **[4]**

Question 5 [27 = 4 + 16 + 3 + 4]

A manufacturer makes three products and uses raw material in limited supply amounts, as shown below in the figure. Each of the 3 products are produced at a separate sub-section of the plant.



Not all of A, B and C have to be totally consumed. There is sufficient market demand for all the available product you produce.

Raw material	Maximum available [kg/day]	Cost [$/kg]
A	4000	1.50
B	3000	2.00
C	2500	2.50

Process	Product	Reactant required per kg product	Operating cost [$/kg]	Selling prices [$/kg]
1	E	0.7 of A and 0.3 of B	$ 0.30 per kg of A consumed	$ 4.00 per kg of E produced
2	F	0.7 of A and 0.3 of B	$ 0.50 per kg of A consumed	$ 3.20 per kg of F produced
3	G	0.4 of A and 0.25 of B and 0.35 of C	$ 0.20 per kg of G *produced*	$ 3.90 per kg of G produced

Operating costs for product F are higher, due to higher electrical heating costs. Please take note of how operating costs are reported.

1. What are your search variables? Describe them, with units, and give them symbols. **[4]**
2. Create the profit objective function, and linear constraints that you will require to solve the optimization problem. Ensure your mathematical problem is written in a natural form, that is interpretable by your engineering colleagues. [*Note*: this does **not** ask for you to write it in standard form; if you intentionally write it in standard form you will be penalized] **[16]**
3. Would this be considered an allocation problem, blending problem, planning problem, or scheduling problem? Explain your answer please. **[3]**
4. Which assumptions might have been made along the way to get this into the desirable linear programming form that you should have achieved in part (2) above? **[4]**

The end.

Newton/Quasi-Newton algorithm

$$f'(x^k) = \frac{f(x^k+h) - f(x^k-h)}{2h}$$

Newton's method step: $\Delta x = x^{k+1} - x^k = -\frac{f'(x^k)}{f''(x^k)}$

$$f''(x^k) = \frac{f(x^k+h) - 2f(x^k) + f(x^k-h)}{h^2}$$

Locating an optimum between 3 points that follow the 3-point pattern. Let the 3 points be x_1, x_2, x_3 and the function values at these points are $f_1 = f(x_1)$, $f_2 = f(x_2)$, $f_3 = f(x_3)$ then the optimum between x_1 and x_3 is:

$$x^* = \frac{1}{2}\left[\frac{(x_2^2 - x_3^2)f_1 + (x_3^2 - x_1^2)f_2 + (x_1^2 - x_2^2)f_3}{(x_2 - x_3)f_1 + (x_3 - x_1)f_2 + (x_1 - x_2)f_3}\right]$$

The line-search problem solves this problem, where x^k is a vector, α is the distance along the search direction, and the search direction is given by this equation, using $+\nabla f(x)$ if we are maximizing and $-\nabla f(x)$ if we are minimizing:

$$\min_{\alpha} \ell(\alpha) \triangleq f(\mathbf{x}^k + \alpha \Delta \mathbf{x}^{k+1})$$

$$\Delta x \triangleq \pm \nabla f(x^k)$$

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Standard form:
$min\ c^T x = c_1 x_1 + c_2 x_2 + ... + c_j x_j + ... + c_n x_n$
$s.t. \quad Ax = b \quad A: m \times n$
$b: m \times 1$
$x: n \times 1$
$max\ f(x) = min\ -f(x)$
Slack variables are added
$\geq b_i$: *by subtraction*
$\leq b_i$: *by addition*
and entries in *b* are all positive.

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Allocation models: to allocate a finite amount of resources.
Blending models: combine resources to best fulfill requirements.
Planning models: decide what actions to take, and where.
Scheduling models: to plan resources to meet varying time demands (the work is already planned out).

marginal value = $\frac{\Delta profit}{\Delta b_i}$

100% rule (used when more than one change is considered)

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The upper limits for the amount of vegetable oil and regular oil that can be supplied are 4500 and 311.1 from the table provided by GAMS.

```
VegSupply                    160.7      200      4500
OilSupply                    11.11      250     311.1
```

Increasing the amount of `OilSupply` by 200 would increase the total amount to 450, above the 311.1 upper limit, a change of basis would occur. Increasing the `VegSupply` by 100 represents a percent change of 2.33% = 100 / (4500-200), increasing the `OilSupply` by 200 represents a percent change of 327.33% = 200 / (311.1-250). Total total change of 2.33 + 327.33 > 100%

Algorithm	**Regular Newton's method**	**Quasi-Newton's method**
Step 0	Chose initial x^0 and and a tolerance ε, and let $k = 0$	Chose initial x^0 , a h value, a tolerance ε, and let $k = 0$
Step 1	Calculate the derivatives $f'(x^k)$ and $f''(x^k)$	Calculate the approximate derivatives $f'(x^k)$ and $f''(x^k)$ from the equations above
Step 2	If $\lvert f'(x^k)\rvert < \varepsilon$ then stop and report x^k as the optimum.	
Step 3	Take the full "Newton step" where $\Delta x = x^{k+1} - x^k = -\frac{f'(x^k)}{f''(x^k)}$ Set $x^{k+1} = x^k + \Delta x$ Set $k \leftarrow k+1$ Repeat from step 1 again	